HAMILTONIAN FORMULATION OF FERROMAGNETIC HYDRODYNAMICS

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Received 10 September 1987; accepted for publication 13 February 1988
Communicated by A.R. Bishop

The equations of ideal ferromagnetic hydrodynamics (FMHD) in three dimensions are formulated as a hamiltonian system in terms of a noncanonical Poisson bracket. The conservation laws for this system are determined and used to construct Lyapunov functionals that are potentially useful for dealing with stability properties of FMHD equilibrium solutions. Classes of equilibrium solutions are identified with critical points of these Lyapunov functionals. The FMHD system is also formulated in n dimensions and the relations among magnetic induction, magnetic field intensity, and magnetization density are discussed from a Lie-algebraic viewpoint.

1. Introduction

We are dealing with the nonlinear hydrodynamics of a compressible fluid that is ideally conducting and is a ferromagnet at the microscopic level. The relation between the magnetic induction $\boldsymbol{B}$ and magnetic field intensity $\boldsymbol{H}$ is arbitrary in

$$\boldsymbol{B}=\boldsymbol{H}+4\pi\boldsymbol{M}, \tag{1}$$

where $\boldsymbol{M}$ is the magnetization density. The problem under consideration for liquid ferromagnets thus differs from the well-known problem of the dynamics of a so-called magnetized liquid, in which the phenomenological relation $\boldsymbol{B}=\mu\boldsymbol{H}$ is introduced, with permeability μ depending on magnetic field, temperature, and density in a prescribed fashion.

Liquid ferromagnets have been observed experimentally and are potentially useful as ideally soft magnetic materials. The unperturbed equilibrium states of liquid ferromagnets are studied in refs. [1–3].

The linear oscillations of ideally conducting ferromagnetic liquids around equilibrium are classified in ref. [4] in the incompressible limit (constant density). These oscillations are shown there to be quite unusual, differing both from the spin waves of a solid ferromagnet and from the magnetohydrodynamics waves (e.g., Alfvén waves) in a conducting liquid in the presence of an external magnetic induction.

In the present work we determine the hamiltonian structure and the resulting conservation laws for the equations of ideal ferromagnetic hydrodynamics (FMHD) in three dimensions. We also provide the generalization of FMHD to n dimensions and discuss the relation (1) among magnetic induction, magnetic field intensity, and magnetization density in the n-dimensional case from a Lie-algebraic viewpoint.

Following ref. [5] the FMHD equations in three dimensions are

$$\operatorname{div} \boldsymbol{B}=0\,, \quad \partial_t \boldsymbol{B}= \operatorname{curl}(\boldsymbol{v}\times\boldsymbol{B})\,, \tag{2a,b}$$

$$\partial_t \rho = -\operatorname{div}(\rho \boldsymbol{v})\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{M}/\rho) = g\rho^{-1}\boldsymbol{M}\times\boldsymbol{H}\,, \tag{3a,b}$$

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -s^2\nabla\rho + \frac{1}{4n}(\operatorname{curl}\boldsymbol{H})\times\boldsymbol{B} + \boldsymbol{M}\times\operatorname{curl}\boldsymbol{H} + (\boldsymbol{M}\cdot\nabla)\boldsymbol{H}\,, \tag{4}$$

where ρ is the mass density, v is the fluid velocity, $s=s(\rho)$ is the sound velocity, $\mathrm{d}/\mathrm{d}t=\partial_t+\boldsymbol{v}\cdot\nabla$ is the material derivative, and g is the gyromagnetic ratio (a constant). Eqs. (1)–(4) form a complete dynamical system for the evolution of the fluid variables $\boldsymbol{B}$, ρ, $\boldsymbol{M}$, and $\boldsymbol{v}$ in a bounded domain subject to the following boundary conditions

$$\boldsymbol{B}\cdot\boldsymbol{n}=0\,, \quad \boldsymbol{v}\cdot\boldsymbol{n}=0\,, \quad \boldsymbol{H}\times\boldsymbol{n}=0\,, \tag{5}$$

where $\boldsymbol{n}$ is the unit vector normal to the boundary. The condition $\operatorname{div}\boldsymbol{B}=0$ is preserved if it is assumed to hold at some initial time.

The physical content of eqs. (2)–(4) can be summarized as follows. According to eq. (2), the divergenceless magnetic induction $\boldsymbol{B}$ is frozen into the fluid. That is, the flux of $\boldsymbol{B}$ is preserved through every surface element comoving with the fluid. By (3a), the mass of each comoving volume element is also preserved. However, by (3b), the specific magnetization $\boldsymbol{M}/\rho$ of a comoving fluid element precesses around the magnetic field at a local gyromagnetic frequency, $g\boldsymbol{H}$. Note that we are treating $\boldsymbol{M}/\rho$ not as a euclidean vector, but as a set of euclidean *scalars* under rotation. The geometrical nature of $\boldsymbol{M}/\rho$ will become clear later, when we discuss the n-dimensional formulation of FMHD. Finally, the motion equation eq. (4) represents momentum balance, or generation of circulation of the fluid by a combination of magnetic-field and magnetization forces.

From eqs. (2), (3) we find

$$\frac{\mathrm{d}m}{\mathrm{d}t}=0\,, \quad m:=\rho^{-1}|\boldsymbol{M}| \quad \text{(specific magnetization)}\,, \tag{6}$$

and

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t}=0\,, \quad \Omega=\rho^{-1}\boldsymbol{B}\cdot\nabla m \quad \text{(potential induction)}\,. \tag{7}$$

That is, m and Ω are invariant along FMHD flow lines. (Actually, $(\rho^{-1}\boldsymbol{B}\cdot\nabla)^N m$ is also a flow-line invariant for any integer power $N\geqslant 0$). Together, eqs. (6), (7), and mass conservation eq. (3a) imply conservation of the global quantity

$$C_\Phi = \int_D \mathrm{d}^3x\,\rho\Phi(m,\Omega) \tag{8}$$

in the domain of flow D for an *arbitrary* function Φ, provided the velocity $\boldsymbol{v}$ is tangential to the boundary ∂D, as specified by condition (5b). The quantity d^3x in (8) is the three-dimensional volume element.

Three other global conservation laws for the FMHD system (1)–(4) also exist. The first of these is the total energy,

$$H = \int_D \mathrm{d}^3x \left(\frac{P^2}{2\rho} + \rho e(\rho) + \frac{1}{8\pi}|\boldsymbol{B}-4\pi\boldsymbol{M}|^2\right), \tag{9}$$

where $\boldsymbol{P}=\rho\boldsymbol{v}$ is the hydrodynamic momentum density and $e(\rho)$ is the specific energy, satisfying

$$\mathrm{d}\left(\rho^2 \frac{\mathrm{d}e}{\mathrm{d}\rho}\right) = \mathrm{d}p = s^2\,\mathrm{d}\rho\,. \tag{10}$$

The second global conservation law is for the magnetic helicity

$$\Lambda = \int_D \mathrm{d}^3x\, \boldsymbol{A}\cdot\boldsymbol{B}\,, \tag{11}$$

where $\boldsymbol{A}$ is the vector potential, satisfying

$$\partial_t \boldsymbol{A} = \boldsymbol{v}\times\boldsymbol{B} - \nabla\Phi\,, \tag{12}$$

where Φ is single-valued in simply connected domains. When the hydrodynamic gauge, $\Phi = \boldsymbol{v}\cdot\boldsymbol{A}$, is chosen, the quantity $\lambda = \rho^{-1}\boldsymbol{A}\cdot\boldsymbol{B}$ (specific helicity) is also a flow-line invariant, $\mathrm{d}\lambda/\mathrm{d}t = 0$, and λ can be added to the arguments of the function Φ in (8), as well as $\rho^{-1}\boldsymbol{B}\cdot\nabla\lambda$, etc. Among these quantities, however, only magnetic helicity (11) is gauge invariant. Finally, the hydrodynamic momentum for FMHD, $\int_D \mathrm{d}^3x\,\boldsymbol{P}$, is conserved, by

$$\partial_t P_i = -[P_i v^j - H_i B^j + (p + H^2/8\pi)\delta_i^j]_{,j} \tag{13}$$

provided the total pressure $(p + H^2/8\pi)$ vanishes on the boundary.

2. Hamiltonian formulation of FMHD

The hamiltonian structure of FMHD consists of the hamiltonian functional given by the energy H in (9) and Poisson bracket defined by

$$\{H, F\} = -\int_D \mathrm{d}^3x \left\{ \frac{\delta F}{\delta P_i}\left((P_j\partial_i + \partial_j P_i)\frac{\delta H}{\delta P_j} + \rho\partial_i \frac{\delta H}{\delta\rho} + (B^j\partial_i - \partial_k B^k \delta_i^j)\frac{\delta H}{\delta B^j} + M_\beta \partial_i \frac{\delta H}{\delta M_\beta}\right)\right.$$
$$\left. + \frac{\delta F}{\delta\rho}\partial_j\rho\frac{\delta H}{\delta P_j} + \frac{\delta F}{\delta B^i}(\partial_j B^i - \delta_j^i B^k\partial_k)\frac{\delta H}{\delta P_j} + \frac{\delta F}{\delta M_\alpha}\left(\partial_j M_\alpha \frac{\delta H}{\delta P_j} + g\epsilon_{\alpha\beta}{}^{\sigma}\frac{\delta H}{\delta M_\beta}\right)\right\}. \tag{14}$$

In the Poisson bracket (14) the operator $\partial_j := \partial(\ )/\partial x^j$ acts to the right on each factor it multiplies. We sum on repeated indices, $i, j, k = 1, 2, 3$ and $\alpha, \beta, \sigma = 1, 2, 3$. The quantity $\epsilon_{\alpha\beta}{}^{\sigma}$ is the totally antisymmetric tensor in three dimensions and represents the structure constants of the Lie algebra SO(3).

By using the hamiltonian H in (9) and the Poisson bracket $\{\ ,\ \}$ defined in (14), the FMHD equations (2)–(4) re-emerge in the form of a hamiltonian system, i.e., as

$$\partial_t F = \{H, F\}\,, \qquad F \in \{\boldsymbol{P}, \rho, \boldsymbol{B}, \boldsymbol{M}\}\,. \tag{15}$$

The variational derivatives of H are given by the formula

$$\delta H = \int_D \mathrm{d}^3x \left(v^j \delta P_j + (e + p/\rho - v^2/2)\delta\rho + \frac{1}{4\pi} H_j \delta B^j - H^\beta \delta M_\beta \right). \tag{16}$$

Note the dual role played in three dimensions by the magnetic field intensity $\boldsymbol{H}$ in (16). We will come back to discuss this when we treat the n-dimensional case, below. For now, substituting the variational derivatives from (16) into the Poisson bracket (14) produces the following dynamical equations,

$$\partial_t \rho = \{H, \rho\} = -(\rho v^j)_{,j}\,, \tag{17a}$$

$$\partial_t B^i = \{H, B^i\} = -(\partial_j B^i - \delta_j^i B^k \partial_k)v^j = [\mathrm{curl}(\boldsymbol{v}\times\boldsymbol{B})]^i - v^i\,\mathrm{div}\,\boldsymbol{B}\,, \tag{17b}$$

$$\partial_t M_\alpha = \{H, M_\alpha\} = (M_\alpha v^j)_{,j} - g\epsilon_{\alpha\beta}{}^{\sigma} M_\sigma H^\beta \,, \tag{17c}$$

$$\partial_t P_i = \{H, P_i\} = -(P_i v^j)_{,j} - P_j v^j_{,i} - \rho(e+p/\rho - v^2/2)_{,i} - \frac{1}{4\pi} B^j (H_{j,i} - H_{i,j}) + \frac{1}{4\pi} H_i \operatorname{div} \boldsymbol{B} + \boldsymbol{M}_\beta H^\beta_i \,. \tag{17d}$$

Eqs. (17a) and (17b) recover the FMHD equations (3a) and (2b) for mass conservation and magnetic induction, provided $\operatorname{div} \boldsymbol{B} = 0$. Using eqs. (17a) and (17b) then converts eqs. (17c) and (17d) into the desired FMHD forms (3b) and (4), respectively. This almost completes the hamiltonian formulation of the FMHD equations. It remains to show that the bilinear, skew-symmetric Poisson bracket (14) satisfies the Jacobi identity,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \quad \forall F, G, H \,. \tag{18}$$

We verify the Jacobi identity by observing that (14) is the natural Poisson bracket on the dual space of the semidirect-product Lie algebra

$$L_1 = D\, \mathrm{s}(\Lambda^0 \oplus \Lambda^2 \oplus (\Lambda^0 \otimes \mathrm{g}))\,, \tag{19}$$

composed of vector fields D on $\mathbb{R}^n$ ($n=3$ in the present case) acting on scalar functions Λ^0, two-forms Λ^2, and functions $\Lambda^0 \otimes \mathrm{g}$ taking values in the Lie algebra denoted g-so(3) (with commutator given by g times the usual so(3) commutator). The symbols s, $\oplus$, and $\otimes$ in (19) denote semidirect product, direct sum, and direct product, respectively. In the semidirect product (19) D acts on itself by commutation of vector fields and acts on differential forms Λ^k, $k=0, 2$, by Lie derivation. Dual coordinates are: $\boldsymbol{P}$ dual to vector fields in D; ρ, to functions in Λ^0; $\boldsymbol{B}$, to two-forms Λ^2; and $\boldsymbol{M}$ to SO(3)-valued functions in $\Lambda^0 \otimes \mathrm{g}$. Since the Poisson bracket is linear in these coordinates, it satisfies the Jacobi identity by virtue of the corresponding identity on the Lie algebra.

For further discussion and examples of Poisson brackets of semidirect-product type appearing in the physics of ideal fluids, see refs. [6–8] and various papers in ref. [9] and refs. [10,11]. When $\boldsymbol{M}$ is absent both in the hamiltonian (9) and in the Poisson bracket (14), we return to ordinary magnetohydrodynamics (MHD) for which the hamiltonian structure was first given in ref. [12] and shown to be dual to a semidirect-product Lie algebra in ref. [6].

Conservation of energy H in (9) now follows from the hamiltonian formulation; by skew-symmetry of the Poisson bracket $\{H, H\} = 0$. Also, conservation of C_Φ in (8) follows from the degeneracy of the Poisson bracket (14), namely

$$\{C_\Phi, F\} = 0 \quad \forall F \in \{\boldsymbol{P}, \rho, \boldsymbol{B}, \boldsymbol{M}\} \,. \tag{20}$$

In particular, C_Φ Poisson-commutes with the hamiltonian H and so is conserved.

Functionals satisfying (20) are said to be Casimirs for the Poisson bracket. In terms of the lagrangian description of FMHD, the Casimirs C_Φ in (20) generate canonical transformations that relabel lagrangian fluid elements, *without* changing the local values of the eulerian dynamical variables $\{\boldsymbol{P}, \rho, \boldsymbol{B}, \boldsymbol{M}\}$. Hence, eq. (20) follows in the eulerian representation of FMHD for any functional F of the eulerian variables.

One immediate observation about such Casimirs is that they can be used to construct Lyapunov functionals for investigating stability of equilibrium solutions. For a description of this method and many examples of Lyapunov stability conditions determined by using Casimirs in ideal fluid and plasma dynamics see ref. [10]. For now, we simply comment that a wide class of FMHD static equilibrium solutions correspond to the critical points of the sum $H_C = H + C_\Phi$. In particular, the equilibrium relation is satisfied at such critical points, namely

$$\partial_t F = \{H_C, F\} = 0 \quad \text{when} \quad \delta H_C = 0 \,. \tag{21}$$

Moreover, the linearized dynamics around such equilibrium solutions that are critical points of H_C preserves the second variation $\delta^2 H_C$; since $\frac{1}{2}\delta^2 H_C$ is the hamiltonian for the linearized dynamics around such critical points. Consequently, those equilibria for which the second variation $\delta^2 H_C$ is of definite sign will be Lyapunov stable (in the norm defined by the quadratic form $\delta^2 H_C$) under the linearized dynamics. See ref. [10] for more details about hamiltonian methods for establishing Lyapunov stability conditions.

3. n-dimensional generalization of FMHD

The Poisson bracket (14) for FMHD in three dimensions can be written in terms of a hamiltonian matrix operator $\mathbf{b}_1$ as

$$\{H, F\} = -\int d^3x \left(\frac{\delta F}{\delta z}\right)^{t} \cdot \mathbf{b}_1 \cdot \frac{\delta H}{\delta z}, \tag{22}$$

where z is the column vector $(\boldsymbol{P}, \rho, \boldsymbol{B}, \boldsymbol{M})^{t}$ and superscript t denotes transpose. Explicitly $\mathbf{b}_1$ for three-dimensional FMHD is given by the skew-symmetric matrix operator,

$$\mathbf{b}_1 = \begin{vmatrix} \partial_j P_i + P_j \partial_i & \rho\partial_i & B^j\partial_i - \partial_k B^k \delta^j_i & M_\beta \partial_i \\ \partial_j \rho & 0 & 0 & 0 \\ \partial_j B^i - \delta^i_j B^k \partial_k & 0 & 0 & 0 \\ \partial_j M_\alpha & 0 & 0 & g\epsilon_{\alpha\beta}{}^{\sigma} M_\sigma \end{vmatrix} \tag{23}$$

The Casimirs C_Φ in (20) satisfy $\mathbf{b}_1 \cdot \delta C_\Phi / \delta z = 0$, i.e. the variational derivatives of C_Φ are null eigenvectors of the hamiltonian matrix $\mathbf{b}_1$.

Now, ordinary MHD without ferromagnetism is a hamiltonian system in an *arbitrary* number of spatial dimensions n [6]. The hamiltonian matrix for MHD in three dimensions in terms of magnetic induction $\boldsymbol{B}$ is simply (23) without the last row and column for the magnetization density. The corresponding hamiltonian matrix for MHD in n dimensions generalizes to

$$\mathbf{b}_2 = -\begin{vmatrix} \partial_j P_i + P_j \partial_i & \rho\partial_i & -B_{jk,i} + \partial_j B_{ik} + \partial_k B_{ji} \\ \partial_j \rho & 0 & 0 \\ B_{mn,l} + B_{ln}\partial_m + B_{ml}\partial_n & 0 & 0 \end{vmatrix} \tag{24}$$

with the corresponding Poisson bracket given by

$$\{H, F\} = -\int d^n x \left[\frac{\delta F}{\delta P_i} \left((\partial_j P_i + P_j \partial_i) \frac{\delta H}{\delta P_j} + \rho \partial_i \frac{\delta H}{\delta p} + (-B_{jk,i} + \partial_j B_{ik} + \partial_k B_{ji}) \frac{\delta H}{\delta B_{jk}} \right) \right.$$
$$\left. + \left(\frac{\delta F}{\delta \rho} \partial_j \rho + \frac{\delta F}{\delta B_{mn}} \left(B_{mn,j} + B_{jn}\partial_m + B_{mj}\partial_n \right) \right) \frac{\delta H}{\delta P_j} \right]. \tag{25}$$

In eqs. (24) and (25) the quantity B_{ij} is a skew-symmetric two-form in $\mathbb{R}^n$ which can be identified with a vector $\boldsymbol{B}$ *only* for $n=3$. The hamiltonian matrix (24) is naturally associated with the dual of the Lie algebra

$$L_2 = D\,\mathrm{s}(\Lambda^0 \oplus \Lambda^{n-2}), \tag{26}$$

with dual coordinates: $\boldsymbol{P}$ dual to vector fields in D; ρ, to functions in Λ^0; and B_{ij}, to $(n-2)$-forms Λ^{n-2}.

The hamiltonian matrix for MHD in n dimensions may also be expressed in terms of the vector potential A_i, $i=1, 2, ..., n$, with $B_{ij} = A_{i,j} - A_{j,i}$. Namely,

$$\mathbf{b}_3 = -\begin{vmatrix} \partial_j P_i + P_j \partial_i & \rho\partial_i & -A_{j,i} + \partial_j A_i \\ \partial_j \rho & 0 & 0 \\ A_{i,j} + A_j \partial_i & 0 & 0 \end{vmatrix}. \tag{27}$$

with corresponding Poisson bracket given by

$$\{H,F\}=-\int \mathrm{d}^n x\left[\frac{\delta F}{\delta P_i}\left((\partial_j P_i+P_j\partial_i)\frac{\delta H}{\delta P_j}+\rho\partial_i\frac{\delta H}{\delta\rho}+(-A_{j,i}+\partial_j A_i)\frac{\delta H}{\delta A_j}\right)\right.$$

$$\left.+\left(\frac{\delta F}{\delta\rho}\partial_j\rho+\frac{\delta F}{\delta A_i}(A_{i,j}+A_j\partial_i)\right)\frac{\delta H}{\delta P_j}\right], \tag{28}$$

which is associated with the Lie algebra

$$L_3=D\,\mathrm{s}(\Lambda^0\oplus\Lambda^{n-1})\,. \tag{29}$$

The map $B_{ij}=A_{i,j}-A_{j,i}$ relates the two hamiltonian matrices (24) and (27). This map, written in the language of differential forms as $B=\mathrm{d}A$, is the natural hamiltonian map generated by the homeomorphism of Lie algebras

$$\mathrm{id}\,\mathrm{s}(\mathrm{id}\oplus(-\mathrm{d})): L_2\to L_3\,. \tag{30}$$

We seek an n-dimensional version of the FMHD equations (1)–(4). It is natural to suppose that the derived equations are also hamiltonian, and it is easy to see that the corresponding hamiltonian structure, which generalizes both the three-dimensional FMHD form (23) and the forms (24) and (27) for n-dimensional MHD is given by the following formulae: in the B_{ij}-representation

$$\mathbf{b}_2'=\left|\begin{array}{c|c} \mathbf{b}_2 & M_\beta\partial_i \\ \hline \partial_j M_\alpha & g t_{\alpha\beta}{}^{\sigma}M_\sigma \end{array}\right|, \tag{31}$$

and in the A-representation

$$\mathbf{b}_3'=\left|\begin{array}{c|c} \mathbf{b}_3 & M_\beta\partial_i \\ \hline \partial_j M_\alpha & g t_{\alpha\beta}{}^{\sigma}M_\sigma \end{array}\right|, \tag{32}$$

where $t_{\alpha\beta}{}^{\sigma}$ are now the structure constants of a Lie algebra $\mathfrak{g}$ ($\mathfrak{g}=\mathrm{so}(3)$ for the three-dimensional case of FHMD (24)) in a basis $\{e_\alpha\}$. The matrices (31) and (32) are naturally associated with the duals of the Lie algebras

$$L_2'=D\,\mathrm{s}(\Lambda^0\oplus\Lambda^{n-2}\oplus(\Lambda^0\otimes\mathfrak{g}))\,, \tag{33}$$

$$L_3'=D\,\mathrm{s}(\Lambda^0\oplus\Lambda^{n-1}\oplus(\Lambda^0\otimes\mathfrak{g}))\,, \tag{34}$$

respectively. Thus, $\boldsymbol{M}=(M_\alpha)\in\Lambda^n\otimes\mathfrak{g}^*$ is now a $\mathfrak{g}^*$-valued density on $\mathbb{R}^n$.

The n-dimensional interpretation of the FMHD hamiltonian H in (9) is slightly less obvious. Clearly, the expression $\boldsymbol{B}-4\pi\boldsymbol{M}$, as it stands, does not make sense in n dimensions; but what is needed is only $|\boldsymbol{B}-4\pi\boldsymbol{M}|^2$ the length squared in H. Thus, we need n-dimensional interpretations of the expressions $|\boldsymbol{B}|^2$, $|\boldsymbol{M}|^2$, and $\boldsymbol{M}\cdot\boldsymbol{B}$. Firstly, there is no problem with the term $|\boldsymbol{B}|^2$: it is simply $\frac{1}{2}B^{ij}B_{ji}$. (Remember, we fixed coordinates on $\mathbb{R}^n$; otherwise we would have to write down a metric on $\mathbb{R}^n$, extend it to $\Lambda^2(\mathbb{R}^n)$, and write $B^2=(B,B)\mathrm{d(vol)}$, where $\mathrm{d(vol)}$ is the volume element formed from the metric.) Secondly, the expression $|\boldsymbol{M}|^2$ only makes sense provided we have a metric on $\mathfrak{g}^*$, which we assume to be nondegenerate and $\mathfrak{g}$-invariant; this is equivalent to having an invariant nondegenerate metric (,) on $\mathfrak{g}$ itself. The requirement of invariance of the metric assures us that the expression $|M|^2=(M,M)=M^\alpha M_\alpha$ is G-invariant with respect to the Lie group G whose Lie algebra is $\mathfrak{g}$. [Here again, we use fixed coordinates on $\mathbb{R}^n$ to decompose M as $\mathrm{d}^n x\otimes\tilde{M}$, where $\tilde{M}=M_\alpha e^\alpha$, so that $|M|^2=\mathrm{d}^n x(\tilde{M},\tilde{M})$.] Thirdly, to make sense out of the expression $\boldsymbol{M}\cdot\boldsymbol{B}$, we assume that we are given: (1) a representation r: $\mathfrak{g}\to\mathrm{End}(\mathrm{V})$, $\mathrm{V}=\mathbb{R}^n$, thus making V and $\mathrm{V}\wedge\mathrm{V}$ into $\mathfrak{g}$-modules, and (2) a homomorphism of $\mathfrak{g}$-modules L: $\mathfrak{g}^*\to\mathrm{V}\wedge\mathrm{V}$. Then extending L naturally into the homomorphism of Λ^0 modules $\Lambda^n\otimes\mathfrak{g}^*\to\Lambda^n\otimes(\mathrm{V}\wedge\mathrm{V})$ via the formula

$$L(\theta v)=\theta L(v)\,,\quad \theta\in\Lambda^n\,,\quad v\in\mathfrak{g}^*\,, \tag{35}$$

we define

$$\boldsymbol{M}\cdot\boldsymbol{B}:=\langle L(M),B\rangle\,,\quad M\in\Lambda^n\otimes\mathfrak{g}^*\,,\quad B\in\Lambda^2(\mathbb{R}^n)=\mathbb{R}^{n*}\wedge\mathbb{R}^{n*}\,. \tag{36}$$

In coordinates, if $L(v_\alpha e^\alpha)=\chi^{\alpha ij}v_\alpha\partial_i\wedge\partial_j$ (where $\partial_i=\partial/\partial x^i$ and we identify $\mathbb{R}^n$ with its tangent space $\mathrm{T}_a(\mathbb{R}^n)$ at any point $a\in\mathbb{R}^n$), and $\boldsymbol{M}=\mathrm{d}^n\boldsymbol{x}\otimes M_\alpha e^\alpha$, $\boldsymbol{B}=B_{ij}\mathrm{d}\boldsymbol{x}^i\wedge\mathrm{d}\boldsymbol{x}^j$, then (36) becomes

$$\boldsymbol{M}\cdot\boldsymbol{B}=\mathrm{d}^n x\,\chi^{\alpha ij}M_\alpha B_{ij}\,, \tag{37}$$

where $\chi^{\alpha ij}$ is a constant. Summarizing, the hamiltonian H for n-dimensional FMHD becomes

$$H=\int\mathrm{d}^n x\left(\frac{1}{2\rho}|\boldsymbol{P}|^2+\rho e(\rho)\right)+\frac{1}{8\pi}\int\mathrm{d}^n x\,B^2+2\pi\int(M,M)-\int\langle L(M),B\rangle\,. \tag{38a,b}$$

(In the A-representation, simply substitute $B=\mathrm{d}A$ in (38b).) For the original case of FMHD in three dimensions, we have $n=3$, $\mathfrak{g}=\mathrm{so}(3)=\mathrm{so}(3)^*\approx\mathbb{R}^3\approx\mathbb{R}^{3*}\approx\mathbb{R}^3\wedge\mathbb{R}^3$, $L=\mathrm{id}$, $\chi^{\alpha ij}=-\frac{1}{2}\epsilon^{\alpha ij}$, the totally antisymmetric tensor, the metric on $\mathfrak{g}\approx\mathbb{R}^3$ coincides with the euclidean metric on $\mathbb{R}^3$, and we recover from (38) the hamiltonian (9).

Remarks. The reader may have wondered why we have not just let L be a linear map, in which case formula (36) would still make sense, but would require in addition that L be a homomorphism of $\mathfrak{g}$-modules. The reason is as follows: this requirement is equivalent to the G-invariance of the expression $\boldsymbol{M}\cdot\boldsymbol{B}$. Indeed, if $\boldsymbol{M}=\theta v$, then for any $h\in\mathfrak{g}$ we have

$$(h.M)\cdot B=\langle L(h.\theta v),B\rangle=\theta\langle L(h.v),B\rangle=\theta\langle h.L(v),B\rangle=-\theta\langle L(v),h.B\rangle=-M\cdot(h.B)\,,$$

where $h.(\;)$ stands for $r(h)(\;)$. In particular, if G acts isometrically on $\mathbb{R}^n$ (so that $\mathfrak{g}$ is forced to be $\mathrm{so}(n)$), then the whole hamiltonian H in (38) is G-invariant.

Since $\mathfrak{g}$ has an invariant metric, the $\mathfrak{g}$-homomorphism L: $\mathfrak{g}^*\to\mathrm{W}:=\mathrm{V}\wedge\mathrm{V}$ uniquely defines (and is uniquely defined by) the $\mathfrak{g}$-homomorphism L^{d}: $\mathfrak{g}\to\mathrm{W}$.

We can now understand the nature of the magnetic field intensity $\boldsymbol{H}$ in general which collapses into $\boldsymbol{H}=\boldsymbol{B}-4\pi\boldsymbol{M}$ for the case $n=3$, $\mathfrak{g}=\mathrm{so}(3)$: in our notation,

$$H=B-\pi(*L(M))\,, \tag{4}$$

where

$$*:\Lambda^n\otimes\Lambda^k(\mathrm{V}^*)\to\Lambda^k(\mathrm{V})\,,\quad \mathrm{V}=\mathbb{R}^n\,, \tag{41}$$

is the density map generated by the metric on $\mathbb{R}^n$.

In ordinary MHD the specific entropy variable η is alos present. To include it, one simply changes $e(\rho)$ in (38a) into $e(\rho,\eta)$, and adds to the hamiltonian matrices $\mathbf{b}'_2$ and $\mathbf{b}'_3$ the extra column and row

$$\begin{array}{c|cc|} & P_j & \\ \hline P_i & & -\eta_{,i} \\ & & \\ & \eta_{,j} & \\ \hline \end{array} \,. \tag{42}$$

This is a particular (cocycleless) case of the basic situation that in quasi-hamiltonian mechanics defines the Lie-algebraic relative Poisson bracket $C^\infty(W^*)\times C^\infty(\mathfrak{g}^*)\to C^\infty(W^*)$ via the formula [13]

$$\{F,f\}(\mu) = -(\mathrm{d}f_{|L^{d})^*(\mu)})\cdot(\mathrm{d}F_{|\mu})\,, \quad F\in C^\infty(W^*)\,, \quad f\in C^\infty(\mathfrak{g}^*)\,, \quad \mu\in W^*\,. \tag{43}$$

Finally, we remark that the extension of the present case from compressible fluids with internal variables interacting with abelian magnetic fields, to the situation of nonabelian Yang–Mills quark–gluon plasma interactions in the magnetohydrodynamic limit is treated in ref. [14].

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